Cotorsion Groups

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Abstract: Cotorsion groups have very important role in the subject of homological algebra. Especially, they have a significant role in studying the functor, the group of extensions. Therefore, this paper makes an attempt to study Cotorsion groups and their properties. Also, a relation between Cotorsion groups and Pure-Injective groups has been obtained. Techniques of homological algebra have been used in proving the main results. Throughout this paper, all the groups are additive Abelian groups and Q is the additive group of rational numbers, Z is the additive group of integers.

Key words: Cotorsion group, Group extensions, Functor

INTRODUCTION

The sequence $E: 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of groups and homomorphisms is called a short exact sequence if (i) α is a monomorphism, (ii) β is an epimorphism, and (iii) $\operatorname{Im}\alpha = \ker\beta$. The group B is called the extension of the group A by the group C. Mac Lane proved that the set of all extensions of a group A by a group C forms a group with respect to the operation of addition of group extensions. The direct sum of two group extensions $E_i: 0 \to A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \to 0,\ i = 1, 2$, is again an extension and is defined as $E_1 \oplus E_2 : 0 \to A_1 \oplus A_2 \xrightarrow{\alpha} B_1 \oplus B_2 \xrightarrow{\beta} C_1 \oplus C_2 \to 0,$ where the circular symbol stands for the direct sum of groups. The diagonal map of the group G is denoted and defined by $\Delta_G : g \mapsto (g, g)$ and the co-diagonal map of the group G is denoted and defined by $\nabla_G : (g_1, g_2) \mapsto g_1 + g_2$. The sum of two extensions E_1, E_2 of a group A by a group C is again an extension $E_1 + E_2 = \nabla_A (E_1 \oplus E_2) \Delta_C$. It is proved that the set of all extensions of the group A by the group C forms a group under the operation of sum of two extensions and it is denoted by $Ext(C, A)$. Also, it is proved that 'Ext' is an additive bi-functor. (refer [1]). The short exact sequence $E: 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is called splitting exact if the group A is a direct summand of the group B. If all the extensions of a group A by a group C are splitting if and only if $Ext(C, A) = 0$. A group D is divisible if $nD = D$, for all integers n. A minimal Divisible group D containing a group G is called a Divisible hull of the group G. All Divisible hulls of a group are isomorphic. If a group A is divisible, then $Ext(C, A) = 0$ for all groups C. A subgroup H of a group G is called a pure subgroup if $nH = H \cap nG$, for all integers n. The short exact sequence $E: 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is said to be a pure exact if the group A is a Pure subgroup of the group B. The set of all pure exact sequences forms a group under the addition of extensions and it is denoted by $Pext(C, A)$. It is obvious that $Pext(C, A) \subset Ext(C, A)$. The set of all homomorphisms from a group A into group C forms a group under the addition of homomorphisims forms a group and it is denoted by $Hom(A, C)$.

MAIN RESULTS

Definition 1: A group G is called a cotorsion group if $Ext(Q, G) = 0$, where Q is the additive group of rational numbers.

Example 1: Divisible groups are Cotorsion.

Towards the properties of the functors 'Ext' and 'Pext' we have

Theorem 2: For the short exact sequence $E: 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ ------- (1), we have

(A) $0 \to Hom(G, A) \to Hom(G, B) \to Hom(G, C) \xrightarrow{E_*} Ext(G, A) \xrightarrow{\beta_*} Ext(G, B) \xrightarrow{\alpha_*} Ext(G, C) \to 0$

(B) $0 \to Hom(C, G) \to Hom(B, G) \to Hom(A, G) \xrightarrow{E^*} Ext(C, G) \xrightarrow{\beta^*} Ext(B, G) \xrightarrow{\alpha^*} Ext(A, G) \to 0$

If the above short exact sequence (1) is pure exact, then

$(C)\quad 0 \to Hom(G,A) \to Hom(G,B) \to Hom(G,C) \xrightarrow{E_*} Pext(G,A) \xrightarrow{\beta_*} Pext(G,B) \xrightarrow{\alpha_*} Pext(G,C) \to 0$

$(D)\quad 0 \to Hom(C,G) \to Hom(B,G) \to Hom(A,G) \xrightarrow{E^*} Pext(C,G) \xrightarrow{\beta^*} Pext(B,G) \xrightarrow{\alpha^*} Pext(A,G) \to 0$

Proof: Refer [1]

Definition 3: A group G is said to be pure-injective if it has the injective property relative to the class of pure exact sequences. That is, for a group G and a for a pure exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, the diagram

$$\begin{array}{ccccccc} 0 \to & A & \xrightarrow{\alpha} & B & \xrightarrow{\phi} & C \to 0 \\ & \phi\downarrow & \swarrow \psi & & & \\ & G & & & & \end{array}$$

is commutative.

Example 4: If the group C is pure-injective, then for every group A, $Hom(A,C)$ is pure-injective.

Lemma 5: A group A is pure-injective if, and only if, $Pext(C,A)=0$ for all groups C.

Proof: Suppose A is pure-injective and it is a pure subgroup of an arbitrary group X. Therefore, the short exact sequence $0 \to A \to X \to X/A \to 0$ is pure exact. Since A is pure-injective, by the definition of pure-injective group, the identity map $I_A : A \to A$ can be factorized as $A \to X \to A$. Thus, A is a direct summand of X. Therefore, the pure exact sequence is splitting exact. Since X is arbitrary, all the pure extensions of A are splitting exact. Hence $Pext(C,A)=0$ for all groups C.

Lemma 6: A group A is pure-injective if, and only if, it is a direct summand of every group X such that A is pure in X and X/A is isomorphic to some torsion-free divisible group or some torsion divisible group.

Proof: Refer [1]

Towards the properties of Cotorsion groups, we have

Lemma 7: Epimorphic image of a cotorsion group is cotorsion.

Proof: Let X be an epimorphic image of a cotorsion group A. Therefore, the sequence $A \to X \to 0$ is exact. For the group of rational numbers Q and by the theorem 2(A) we have, an exact sequence $Ext(Q,A) \to Ext(Q,X) \to 0$. Since, A is cotorsion, the first group is zero and hence the second group $Ext(Q,X)=0$ which implies the group X is cotorsion.

Lemma 8: A group is cotorsion if and only if it is an epimorphic image of a pure-injective group.

Proof: Since any group can be embedded in divisible group, let the group A be embedded in a divisible group X so that the sequence $0 \to A \to X \to X/A \to 0$ is short exact. By theorem 2(A), this sequence induces the exact sequence $Hom(Q/Z, X/A) \to Ext(Q/Z, A) \to Ext(Q/Z, X)$ in which the last group is zero, since X is divisible. Since the group X/A is divisible, the first group is pure-injective. Thus, the second group is an epimorphic image of a pure-injective group and hence it is cotorsion.

Theorem 9: The group $Ext(C,A)$ is cotorsion for all the groups A, C.

Proof: Follows from the proof of Lemma 8, if we consider any group C in the place of the group Q/Z.

The following theorem reveals the relation between the pure-injective group and a cotorsion group.

Theorem 10: A cotorsion group A is pure-injective if, and only if, $Pext(Q/Z, A)=0$.

Proof: The proof follows from Lemmas 5 and 6.

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